AD-A095 658 CONNECTICUT UNIV STORRS DEPT OF ELECTRICAL ENGINEERI--ETC F/G 17/2
BOUNDS ON TRANSMISSION DELAYS FOR THE CCRA.(U)
JAN 81 L GEORGIADIS, P PAPANTONI-KAZAKOS AFOSR-78-3695
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SCHOOL OF ENGINEERING
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BOUNDS ON TRANSMISSION DELAYS
FOR THE CCRA

Leonidas Georgiadis

Technical Report EECS TR-81-2

January, 1981

SECURITY CLASSIFICATION OF THIS PAGE: UNCLASSIFIED

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: AFOSR-TR- 81-0133	2. GOVT ACCESSION NO.: AD-A095 658
3. RECIPIENT'S CATALOG NUMBER	
4. TITLE (and Subtitle): BOUNDS ON TRANSMISSION DELAYS FOR THE CCRA	5. TYPE OF REPORT & PERIOD COVERED: Interim
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s): Leonidas Georgiadis; P. Papantoni-Kazakos	8. CONTRACT OR GRANT NUMBER(s): AFOSR-78-3695; NSF-ECS-79-08825
9. PERFORMING ORGANIZATION NAME AND ADDRESS: University of Connecticut, Department of Engineering and Computer Science, Storrs, CT. 06268	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: 61102F 2304/A5
11. CONTROLLING OFFICE NAME AND ADDRESS: Air Force Office of Scientific Research/NM, Bolling AFB, Washington, DC 20332	12. REPORT DATE: January 1981
	13. NUMBER OF PAGES: 29
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office): UCT-TR-81-2; Technical report	15. SECURITY CLASS. (of this report): UNCLASSIFIED
	15a. DECLASSIFICATION DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

In this paper, we are concerned with the per packet transmission delays induced by the Capetabakis collision resolution protocol for infinitely large number of identical bursty users. We first correct the existing lower and upper bounds on the expected per packet transmission delay E(D). Then, we proceed by developing a new upper bound on E(D). This new bound is much tighter approaching the lower bound for large arrival rates.

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

BOUNDS ON TRANSMISSION DELAYS FOR THE CCRA

Leonidas Georgiadis

P. Papantoni-Kazakos

acting as advisor and editor

The University of Connecticut

Department of Electrical Engineering and Computer Science

U-157

Storrs, Connecticut 06269

This work was supported jointly by the National Science Foundation under the Grant ECS-79-08815, and the Air Force Office of Scientific Research under Grant AFOSR-78-3695.

Abstract

In this paper, we are concerned with the per packet transmission delays induced by the Capetanakis collision resolution protocol for infinitely large number of identical bursty users.

We first correct the existing lower and upper bounds on the expected per packet transmission delay $E\{D\}$.

Then, we proceed by developing a new upper bound on $E\{D\}$. This new bound is much tighter, approaching the lower bound for large arrival rates.

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1. Introduction

In this paper, we are concerned with the "multiple-access" problem, where a large number of independent, packet transmitting, bursty users request access to a common channel. We consider "random-access" transmission protocols, as more efficient for the present problem [4].

In more specific terms, the user and channel models considered, are as follows:

(i) The users are independent from each other, and they can communicate with each other only through the channel. Furthermore, the users are large in number, they are identical and bursty, and each generates packets of fixed common length. The cumulative input to the channel traffic is a Poisson process.

(ii) The common channel is perfect, i.e. there are no channel errors. In addition, the channel time is divided in slots of identical length, where this length is equal to the length of one packet.

(iii) The feedback channel is perfect, i.e. it does not induce propagation delays. Furthermore, the feedback channel broadcasts with no errors the outcome from each channel slot. In particular, it broadcasts a trenary sequence, where the value of each digit from this sequence indicates if the corresponding slot was empty, busy with exactly one packet, or busy with at least two packets.

(iv) The transmission characteristics imply low-level synchronization among the users. In particular, each user is allowed to transmit at most one packet at the time, and he can start transmission only at the beginning of some channel slot. Thus, some channel slot is empty if no user transmitted a packet within it. Some channel slot is busy with exactly one packet if exactly one user transmitted within it. It is assumed that in this last case the transmitted packet is received correctly. Incorrect transmission occurs only if at least two packets are transmitted within the same slot. Then, collision occurs, and the involved packets are lost completely. If so, those packets are restored in the queues of the corresponding users, and are retransmitted within some future slots.

Any "random-access" transmission protocols, which are appropriate for the above model, are characterized by a number of performance parameters. The number one such performance parameter, which in fact determines the eligibility of a given random-access transmission protocol, is stability. A stable random-access transmission protocol maintains the rate of the cumulative input Poisson traffic, while an unstable such protocol does not. Given a stable random-access transmission protocol, two other performance characterisitcs for comparison with other stable such protocols, are the throughput and the per packet transmission delay. The throughput is defined as the ratio $\rho = \frac{\#\text{successful transmissions}}{\#\text{transmission attempts}}$ induced by the random-access transmission protocol, and it is related to the channel capacity. In fact, the channel capacity is the maximum of all throughputs induced by stable random-access transmission protocols. The per packet transmission delay is defined as the time between the arrival of some packet and its successful transmission, where time is measured in number of channel slots.

The oldest existing random-access transmission protocol for the user and channel models stated in this paper, is the slotted-Aloha. The problems regarding the operation of the slotted-Aloha, are by now well known [1-4], and we will not discuss them here. Instead, we will focus on the protocol by Capetanakis [1-4]. Capetanakis' protocol has been called collision-resolution protocol, it is stable for input Poisson rates below .3465, and it induces a throughput of .43. Refinements of Capetanakis' algorithm by Gallager, Massey, and Mosely, have increased the input Poisson rate for stability to .375, and the induced throughput to .488 [4].

In this paper, we concentrate on the per packet transmission delay induced by Capetanakis' collision resolution protocol. Afterall, the very reason for consideration of random-access protocols is the improvement of delays.

Capetanakis studied the per packet transmission delay (or waiting time) induced by his otocol. The resulting expression being complex, a lower and an upper bound on this delay were developed. The procedure for the derivation of these bounds can be found in [1,4].

Here, we develop tighter lower and upper bounds on the per packet transmission delays induced by Capetanakis' protocol, through the correction of some step in the used procedure [1,4] first, and then through a fresh approach.

2. The Capetanakis Protocol - Notation

We will use reference [4], since we feel that Capetanakis' protocol and its analysis are best explained there. We will use basically the same notation as in [4], and we will describe the Capetanakis Collision Resolution Protocol (CCRA) only briefly and quantitavely.

The CCRA is activated just after a collision slot. Then, through the feedback channel, all users are instructed to withhold newly generated packets until the collision is resolved. The collision resolution interval (CRI) is the number of slots required for the resolution of the collision, that is the successful transmission of all packets involved in the collision. The collision resolution is obtained through the application of the CCRA, during the CRI. The CCRA is based on the following general principle:

After each collision slot within the CRI, each user involved in this collision flips independently a binary fair coin with outcomes 0 and 1. Among those users, only the ones with outcome 0 transmit within the next slot. Until the initial collision is completely resolved, the resolution of no other collision within the CRI is attempted.

The CCRA has the structure of a binary tree, where each node within the tree is taken to the leaves-depth, before other nodes on the same depth with the original node are resolved.

Using the reasonable for infinite number of identical bursty users assumption, that during any CRI there may be at most one new packet arrival per user, we define as in [4] the following parameters:

L_N : The expected length of a CRI, given that the number of packets involved in the collision within the slot with which the CRI starts, is equal to N.

X_i : The number of packets involved in the collision of the first slot of the ith CRI, from the beginning in time that the system starts operating.

Y_i : The length of the ith CRI.

X_∞ : The X_i for $i\to\infty$, that is in steady-state.

Y_∞ : The Y_i for $i\to\infty$.

Y_a : In steady-state, the length of the CRI in progress, when some new packet arrival occurs.

Y_d : In steady-state, the length of the CRI in progress, when some packet departs from the system (that is, when the packet is successfully transmitted).

X_d : In steady-state, the number of packets involved in the collision of the first slot of the CRI, during which a packet departs from the system.

D : In steady-state, the time interval measured in number of slots, from the arrival of some packet to its departure from the system.

λ : The rate of the cumulative input Poisson process.

$$\delta_{iN} = \begin{cases} 1 \ ; \ \text{if } i = N \\ 0 \ ; \ \text{otherwise} \end{cases}$$

We will point out here that the existence of steady-state for the CCRA has been proven rigorously by Capetanakis.

Using the notation presented in the present section, in the next section we will outline the approach taken and the results obtained in [4], regarding the per packet transmission delays induced by the CCRA.

3. The CCRA Transmission Delays

In [4], lower and upper bounds on the expected per packet transmission delays (waiting times) E{D} induced by the CCRA, have been obtained. The bounds are valid for rates λ in the region [0, .3465]. Within this λ region, the expected length of each CRI is finite, and the system reaches steady-state.

The following approach was taken in the development of the bounds (in [4]).

First a recursive expression for the evaluation of L_N was obtained. Based on this expression, the following lower and upper bounds on L_N were obtained:

$$L_N \geq 2.8810\, N - 1 + 2\, \delta_{0N} - 0.8810\, \delta_{1N} \tag{1}$$

$$L_N \leq 2.8867\, N + \delta_{0N} - 1.8867\, \delta_{1N} \tag{2}$$

In parallel, it was also found that the following equality holds:

$$E\{Y_a\} = \frac{E\{Y_\infty^2\}}{E\{Y_\infty\}} \tag{3}$$

From this point on, the key expression used for the eventual evaluation of the bounds for E{D}, was the following:

$$E\left\{ X_d \middle/ Y_a = L \right\} = \lambda L \tag{4}$$

Expression (4) is true due to the fact that the arrival process is Poisson, and it results in the following expression:

$$E\{X_d\} = \lambda\, E\{Y_a\} \tag{5}$$

The bounds in (1) and (2), in conjuction with expression (5) and the obvious relation $E\left\{ Y_d \middle/ X_d = N \right\} = L_N$, result then in the following lower and upper bounds for the expected value $E\{Y_d\}$:

$$E\{Y_d\} \geq 2.8810\, \lambda\, E\{Y_a\} - 1.8810 + 2.8810\, e^{-\lambda\, E\{Y_a\}} \tag{6}$$

$$E\{Y_d\} \leq 2.8867\, \lambda\, E\{Y_a\} + (1 - 1.8867\lambda)\, e^{-\lambda} \tag{7}$$

The bounds in (6) and (7) are expressed as functions of the rate λ and the expected value $E\{Y_a\}$.

Subsequently the following lower and upper bounds on the expected delay E{D} were found, in terms of $E\{Y_a\}$ and $E\{Y_d\}$:

$$\frac{1}{2}\, E\{Y_a\} + \frac{1}{2}\, E\{Y_d - 1\} \leq E\{D\} \leq \frac{1}{2}\, E\{Y_a\} + E\{Y_d - 1\} \tag{8}$$

The bounds in (8), in conjuction with the bounds in (6) and (7), result clearly in the following bounds for E{D}*:

$$E\{D\} \geq \left(\frac{1}{2} + 1.4405\, \lambda\right) E\{Y_a\} - 1.4405 \left(1 - e^{-\lambda\, E\{Y_a\}}\right) \tag{9}$$

$$E\{D\} \leq \left(\frac{1}{2} + 2.8867\, \lambda\right) E\{Y_a\} + (1 - 1.8867\lambda)\, e^{-\lambda} - 1 \tag{10}$$

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*In expressions (4.42) and (4.43) in [4], a parenthesis is missing.

The bounds in (9) and (10) are functions of λ and $E\{Y_a\}$ only.

Finally, tight bounds on $E\{Y_\infty\}$ and $E\{Y_\infty^2\}$, result in tight lower and upper bounds for $E\{Y_a\}$ in (3). These bounds clearly provide lower and upper bounds for $E\{D\}$ through (9) and (10), and they are given by the following expressions:

$$E\{Y_a\} \le \begin{cases} \dfrac{5.964\lambda + 1 - 5.964\lambda(1-\lambda)(1 - 2.8867\lambda)/[1 - 2.8867\lambda(1-\lambda)]}{1 - 8.333\lambda^2} ; & \lambda \le .22 \\ \dfrac{5.964\lambda + 1}{1 - 8.333\lambda^2} ; & .22 < \lambda < .3464 \end{cases} \tag{11}$$

$$E\{Y_a\} \ge \begin{cases} \dfrac{5.897\lambda + (1 - 5.897\lambda)(1 - 2.8867\lambda)/[1 - 2.8867\lambda(1-\lambda)]}{1 - 8.300\lambda^2} ; & \lambda \le .1696 \\ \dfrac{1}{1 - 8.300\lambda^2} ; & .1696 < \lambda < .3464 \end{cases} \tag{12}$$

The bounds in (11) and (12), in conjuction with the bounds in (9) and (10), provide lower and upper bounds for $E\{D\}$, which are functions of the rate λ only, as desired.

These bounds have been computed for different λ values [4]. We will discuss the results from the computations in [4] as they compare to our results, in the following section.

4. Modified E{D} Bounds

In section 3, we outlined the approach taken in [4] for the computation of lower and upper bounds on the expected per packet transmission delay $E\{D\}$.

The key expression for the computation of the $E\{D\}$ bounds in [4], was expression (4) in section 3 of the present paper. But the assumption throughout the whole related analysis, is that there exists at least one packet arrival within the Y_a

slot period. Thus, the relevant key expression in the computation of the E{D} bounds should be an expression for the expectation $E\{X_d | Y_a = L,\ X_d \geq 1\}$, rather than the expression for the expectation $E\left\{X_d / Y_a = L\right\}$ in (4).

In this section, we will substitute expression (4) by an expression for the expectation $E\left\{X_d / Y_a = L,\ X_d \geq 1\right\}$, and we will subsequently compute modified lower and upper bounds for the expected value E{D}.

Due to the fact that the packet arrival process is Poisson with rate λ, we clearly have:

$$P\left(X_d = k / Y_a = L,\ X_d \geq 1\right) = \begin{cases} 0 & ;\ k = 0 \\ \dfrac{P\left(X_d = k / Y_a = L\right)}{P\left(X_d \geq 1 / Y_a = L\right)} = \dfrac{e^{-\lambda L}\dfrac{(\lambda L)^k}{k!}}{1 - e^{-\lambda L}} & ;\ k > 0 \end{cases} \tag{13}$$

From expression (13), we have in a straight-forward manner:

$$E\left\{X_d / Y_a = L,\ X_d \geq 1\right\} = \frac{\lambda L}{1 - e^{-\lambda L}} \tag{14}$$

$$P\left(X_d = k / X_d \geq 1\right) = E\left\{\frac{e^{-\lambda Y_a}\dfrac{(\lambda Y_a)^k}{k!}}{1 - e^{-\lambda Y_a}} \Bigg/ X_d \geq 1\right\} \tag{15}$$

; where the expectation in (15) is with respect to Y_a.

At this point, let us observe that the implicit assumption is that at least one arrival occurred during the slot interval represented by Y_a. This assumption implies $X_d \geq 1$. It is due to this observation that Massey [4] dropped the conditioning $X_d \geq 1$ from the expectation $E\left\{X_d \big/ Y_a = L,\ X_d \geq 1\right\}$. Unfortunately, however, the above observation does not imply that the X_d process is Poisson. The distribution of X_d is rather reflected by expression (14).

Using the above observation, we will delete from now on the conditioning $X_d \geq 1$ from expressions involving expectations on Y_a. Then, the $X_d \geq 1$ conditioning in (15) is deleted, and we proceed as follows.

From expression (15) we obtain directly the following expressions:

$$P\left(X_d = 1 \big/ X_d \geq 1\right) = E\left\{\frac{\lambda Y_a\, e^{-\lambda Y_a}}{1 - e^{-\lambda Y_a}}\right\} \tag{16}$$

$$E\left\{X_d \big/ X_d \geq 1\right\} = E\left\{\frac{\lambda Y_a}{1 - e^{-\lambda Y_a}}\right\} \tag{17}$$

In expressions (16) and (17) the expectations are with respect to Y_a.

Now, we will use the upper bound for L_N, as given by expression (2), where the term δ_{0N} is deleted due to the $X_d \geq 1$ conditioning. We obtain then, the following inequality

$$E\left\{Y_d \big/ X_d = N,\ X_d \geq 1\right\} = L_N \leq 2.8867\ N - 1.8867\ \delta_{1N}\ ;\ N \geq 1 \tag{18}$$

Directly from (18), we also obtain the following expression.

$$E\left\{Y_d \big/ X_d \geq 1\right\} \leq 2.8867\ E\left\{X_d \big/ X_d \geq 1\right\} - 1.8867\ P\left(X_d = 1 \big/ X_d \geq 1\right) \tag{19}$$

Substituting expressions (16) and (17) in expression (19), we obtain:

$$E\left\{Y_d \big/ X_d \geq 1\right\} \leq 2.8867\ E\left\{\frac{\lambda Y_a}{1 - e^{-\lambda Y_a}}\right\} - 1.8867\ E\left\{\frac{\lambda Y_a\, e^{-\lambda Y_a}}{1 - e^{-\lambda Y_a}}\right\} \tag{20}$$

In the right part of inequality (20), we add and subtract the term $E\left\{\frac{\lambda Y_a\, e^{-\lambda Y_a}}{1 - e^{-\lambda Y_a}}\right\}$. We then obtain:

$$E\left\{ Y_d \middle/ X_d \geq 1 \right\} \leq 2.8867\ \lambda\ E\{Y_a\} + E\left\{\frac{\lambda Y_a}{e^{\lambda Y_a} - 1}\right\} \tag{21}$$

Observing now that $Y_a \geq 1$, and that for $x > 0$ the function $\frac{\lambda x}{e^{\lambda x} - 1}$ is monotonically decreasing with increasing x, and it is convex for $x \geq 1$, we can obtain the following bound:

$$E\left\{\frac{\lambda Y_a}{e^{\lambda Y_a} - 1}\right\} \leq \frac{\lambda}{e^{\lambda} - 1} \quad ; \quad \text{For } Y_a \geq 1 \tag{22}$$

Substituting the bound in (22) in expression (21), we obtain:

$$\boxed{E\left\{ Y_d \middle/ X_d \geq 1 \right\} \leq 2.8867\ \lambda\ E\{Y_a\} + \frac{\lambda}{e^{\lambda} - 1}} \tag{23}$$

Expression (23) provides an upper bound on the expectation $E\left\{ Y_d \middle/ X_d \geq 1 \right\}$, which is different than the upper bound in (7), if the conditioning $X_d \geq 1$ is deleted as necessary and implicit.

Using the lower bound for L_N, as given by expression (1), we can derive similarly a lower bound for $E\left\{ Y_d \middle/ X_d \geq 1 \right\}$.

Indeed, deleting again the term δ_{ON} in (1), we obtain:

$$E\left\{ Y_d \middle/ X_d = N,\ X_d \geq 1 \right\} = L_N \geq 2.8810\ N - 1 - 0.8810\ \delta_{1N} \tag{24}$$

Averaging out with respect to N, as in (19), and substituting expressions (16) and (17), we obtain from (24):

$$E\left\{ Y_d \middle/ X_d \geq 1 \right\} \geq 2.8810\ E\left\{\frac{\lambda Y_a}{1 - e^{-\lambda Y_a}}\right\} - 1 - 0.8810\ E\left\{\frac{\lambda Y_a\, e^{-\lambda Y_a}}{1 - e^{-\lambda Y_a}}\right\} \tag{25}$$

Adding and subtracting the term $2E\left\{\frac{\lambda Y_a\, e^{-\lambda Y_a}}{1-e^{-\lambda Y_a}}\right\}$ in the right part of (25), we obtain:

$$E\left\{{Y_d}\middle/{X_d \geq 1}\right\} \geq 2.8810\ \lambda\ E\{Y_a\} - 1 + 2\ E\left\{\frac{\lambda Y_a}{e^{\lambda Y_a}-1}\right\} \qquad (26)$$

Due to the convexity of the function $\frac{\lambda x}{e^{\lambda x}-1}$ for $x \geq 1$, and due to the fact that $Y_a \geq 1$, applying the Jensen inequality we obtain the following bound:

$$E\left\{\frac{\lambda Y_a}{e^{\lambda Y_a}-1}\right\} \geq \lambda\ \frac{E\{Y_a\}}{e^{\lambda E\{Y_a\}}-1} \qquad (27)$$

Substituting the bound in (27) to the inequality in (26), we finally obtain:

$$\boxed{E\left\{{Y_d}\middle/{X_d \geq 1}\right\} \geq 2.8810\ \lambda\ E\{Y_a\} - 1 + 2\ \frac{\lambda E\{Y_a\}}{e^{\lambda E\{Y_a\}}-1}} \qquad (28)$$

Expression (28) provides a lower bound on the expectation $E\left\{{Y_d}\middle/{X_d \geq 1}\right\}$, which is different than the lower bound in (6), if the conditioning $X_d \geq 1$ is deleted as necessary and implicit.

Using now the bounds in (23) and (28), in expression (8), after deleting the conditioning $X_d \geq 1$, we obtain the following modified bounds for the expectation $E\{D\}$:

$$E\{D\} \geq \left(\frac{1}{2} + 1.4405\ \lambda\right) E\{Y_a\} - 1 + \frac{\lambda E\{Y_a\}}{e^{\lambda E\{Y_a\}}-1} \qquad (29)$$

$$E\{D\} \leq \left(\frac{1}{2} + 2.8867\ \lambda\right) E\{Y_a\} - 1 + \frac{\lambda}{e^{\lambda}-1} \qquad (30)$$

The bounds in (29) and (30) are different than the corresponding bounds in (9) and (10), and they are functions of the rate λ and the expected value $E\{Y_a\}$.

If the bounds on the expected value $E\{Y_a\}$, as given by expressions (11) and (12), are used in expressions (29) and (30), modified lower and upper bounds on $E\{D\}$ can be obtained. These last bounds are functions of the rate λ only, as desired.

We performed parallel computations for the bounds in (9) and (10) and the bounds in (29) and (30), for different values of λ. Our results are exhibited in table 1. As expected, due to the correction we obtained by the addition of the conditioning $X_d \geq 1$, our modified lower and upper bounds are higher than the corresponding bounds in [4], for all λ values. Also, as expected, the correction has stronger effects on low λ values. In fact, for λ values below 0.15, our modified lower bound is higher than the upper bound in [4]. In general, the curves indicating the modified lower and upper bounds are shifted upward versions of the corresponding curves in figure 4.1 of [4].

λ	Lower Bounds		Upper Bounds	
	O	M	O	M
0.0500	0.521	0.569	0.531	0.645
0.1000	0.599	0.701	0.664	0.881
0.1500	0.780	0.961	1.009	1.319
0.1696	0.902	1.125	1.252	1.595
0.2000	1.200	1.505	1.842	2.236
0.2500	3.423	3.958	5.762	6.231
0.3000	9.009	9.521	14.560	15.096
0.3333	38.067	38.508	58.204	58.781
0.3400	80.203	80.643	121.458	122.043
0.3450	371.910	372.351	559.442	560.032

O : Bounds in [4]

M : Modified Bounds in (29) and (30).

Table 1

5. Improved Bounds for E{D}

In section 4, we simply corrected the bounds in [4] for E{D}.

In this section, we will take a fresh approach, to develop tighter bounds on E{D}.

We will first develop an exact expression for a per packet waiting time parameter. We will then present an intuitive analysis and subsequent bounds on this parameter. Finally, using these bounds, we will compute bounds on E{D}.

Given that N packets are involved in the collision within the first slot of some CRI, we will first seek expressions for the expected transmission delay for each of the N packets, within the CRI. We will adopt Capetanakis' collision resolution protocol [1 - 4].

After the first collision slot within the CRI, which started with N collided packets, let i out of the N users have outcome 0 from flipping their fair coin. Then, the corresponding outcome for the remaining N-i users is 1. Let us denote this event (i, N-i). Then, given the event (i, N-i), each of the N packets gives outcome 0, with probability $\frac{i}{N}$. It gives outcome 1, with probability $\frac{N-i}{N}$.

Define:

d_j^i ; $j = 0,1$: The expected delay in the transmission of some packet within the CRI, given that the packet gave outcome j at the first trial, and that the first trial event is (i, N-i).

$m_{N/i}$: The expected delay in transmission of some packet within the CRI, given that N packets were involved in the collision within the first slot of the CRI, and that the first trial event is (i, N-i).

m_N : The expected delay in transmission of some packet within the CRI, given that N packets were involved in the collision within the first slot of the CRI.

Then, we have:

$$m_{N/i} = d_0^i \frac{i}{N} + d_1^i \frac{N-i}{N} \tag{31}$$

Also, defining $m_0 = 0$, observing that $m_1 = 0$, and considering the operation of the CCRA, we can easily see that the following expressions hold for $N \geq 2$:

$$\begin{aligned} d_0^i &= 1 + m_i \\ d_1^i &= 1 + L_i + m_{N-i} \end{aligned} \tag{32}$$

; where L_i is defined in section 2 of this paper. Substituting expressions (32) in (31) we obtain:

$$m_{N/i} = 1 + m_i \frac{i}{N} + \frac{N-i}{N} (m_{N-i} + L_i) \quad ; \quad N \geq 2 \tag{33}$$

Clearly, the probability with which the event (i, N-i) occurs is $\binom{N}{i} \frac{1}{2^N}$. Therefore, we have:

$$m_N = \sum_{i=0}^{N} \binom{N}{i} \frac{1}{2^N} m_{N/i} \tag{34}$$

Substituting expression (33) in expression (34), we obtain:

$$(2^N - 2) m_N = 2^N + 2 \sum_{i=1}^{N-1} \binom{N-1}{i-1} m_i + \sum_{i=0}^{N-1} \binom{N-1}{i} L_i \tag{35}$$

Now, we use the following expressions from [4]:

$$\begin{aligned} 2^N L_N &= 2^N + 2 \sum_{i=0}^{N} \binom{N}{i} L_i \quad ; \quad N \geq 2 \\ L_0 &= L_1 = 1 \end{aligned} \tag{36}$$

Using expressions (36) and (35), we finally obtain the following exact recursive expressions for m_N, which also involve the L_i's:

$$\boxed{\begin{aligned} &(2^N - 2)\, m_N = 2^{N-2} L_{N-1} + 2\sum_{i=1}^{N-1} \binom{N-1}{i-1} m_i + 3(2^{N-2}) \ ; \ N \geq 3 \\ &m_0 = m_1 = 0 \\ &m_2 = 3 \end{aligned}} \tag{37}$$

In the appendix, we present an alternative procedure for the exact evaluation of the m_N parameters.

Unfortunately, the expressions in (37) are complex, and the development of bounds for the m_N's from them, seems impossible. For that reason, we will use some intuitive analysis for the evaluation of the m_N's. Then, we will verify the results from the intuitive analysis, by computationally comparing with the exact expressions in (37).

a. Intuitive Analysis for m_N

Consider some CRI, which starts with a slot with 2N collided packets. If $N \to \infty$, then with probability close to one, the first trial will result in N outcomes equal to 0, and N outcomes equal to 1. Thus,

$$\underset{N\to\infty}{m_{2N}} = \begin{cases} 1 + m_N \ ; \text{ with probability} \sim \frac{1}{2} \\ 1 + L_N + m_N \ ; \text{ with probability} \sim \frac{1}{2} \end{cases} \tag{38}$$

Therefore, we conclude from (38) that for $N \to \infty$ we have:

$$m_{2N} \simeq 1 + m_N + L_{N/2} \tag{39}$$

Following the recursions induced by expression (39), we obtain:

$$m_{2^k} \simeq k + m_1 + \frac{1}{2}\sum_{i=1}^{k-1} L_{2^i} \text{ ; for N power of 2, and N large} \tag{40}$$

Similarly, for N equal to a power of 2, and N large enough, we also obtain:

$$L_{2^k} \simeq k + L_1 + \sum_{i=1}^{k-1} L_{2^i} \tag{41}$$

Substituting (41) in (40), we then obtain:

$$m_{2^k} \simeq \frac{k}{2} + \frac{L_{2^k}}{2} - \frac{L_1}{2} + m_1 \tag{42}$$

Denoting $N = 2^k$ and $k = \ell og_2 N$, we finally obtain from (42) the following approximation:

$$m_N \simeq .5\ \ell og_2 N + \frac{L_N}{2} + C \quad ; \quad N \to \infty \tag{43}$$

; where C is a variable equal to $m_1 - \frac{L_1}{2}$, whose influence on the expression (43) is infinitesimal for large N's.

Let us now define:

$$\boxed{m_N^{\alpha} = .5\ \ell og_2 N + \frac{L_N}{2}} \tag{44}$$

Then, m_N^{α} provides an approximation for the parameter m_N, whose exact expression is given by (37).

It is unknown, at this point, how good an approximation of m_N, m_N^{α} is. We will study the closeness between m_N and m_N^{α} numerically. We present our results on that, in the following subsection.

b. Comparison between m_N and m_N^α

We denote by m_N the exact expression in (37). We denote by m_N^α the approximation in (44). We define:

$$e_N = m_N - m_N^\alpha \tag{45}$$

We study the closeness between the expressions for m_N and m_N^α numerically, by computing the parameters e_N and $N^{-1}e_N$ for different N's. The results of our computations are exhibited in table 2.

From the results in table 2, we observe that for increasing N, the parameter e_N increases monotonically, while the rate of increase $N^{-1}e_N$ decreases monotonically. In fact, we observe that for $N \geq 2$, the $N^{-1}e_N$ values remain within the interval [0, .019]. Due to the monotonicity of the function $f(N) = N^{-1}e_N$, the observed upper bound .019 on f(N) is pessimistic for N values larger than 70. Using this pessimistic bound for large N's, together with the lower bound 0 for f(N) and with expressions (44) and (45), we obtain the following lower and upper bounds on the exact m_N, for $N \geq 1$:

$$.5\ \ell og_2 N + \frac{L_N}{2} - .5\ \delta_{1N} \leq m_N \leq .5\ \ell og_2 N + \frac{L_N}{2} + .019N - .5\ \delta_{1N} \tag{46}$$

The bounds in (46) will be used for the computation of lower and upper bounds for the expectation E{D}.

c. Lower and Upper Bounds on E{D}

We will first proceed with bounds for the expectation $E\{m_N\}$. These bounds will be found through the computation of the expected values of the bounds in (46), conditioned on $X_d \geq 1$ (as in section 4 of this paper).

We will first compute the expected values of each term in the bounds appearing in (46). We recall that the ramdon variable N corresponds to $X_d = N$, where X_d is

N	m_N	m_N^{α}	$N^{-1} e_N$	e_N
1	0.0	0.500000D+00	-0.500000D+00	-0.500000D+00
2	0.300000D+01	0.300000D+01	0.222045D-15	0.444089D-15
3	0.466667D+01	0.462581D+01	0.136174D-01	0.408521D-01
4	0.633333D+01	0.626190D+01	0.178571D-01	0.714286D-01
5	0.796190D+01	0.787049D+01	0.182834D-01	0.914169D-01
6	0.955361D+01	0.944901D+01	0.174334D-01	0.104600D+00
7	0.111179D+02	0.110041D+02	0.162474D-01	0.113732D+00
8	0.126631D+02	0.125427D+02	0.150531D-01	0.120424D+00
9	0.141950D+02	0.140695D+02	0.139529D-01	0.125576D+00
10	0.157173D+02	0.155876D+02	0.129691D-01	0.129691D+00
11	0.172319D+02	0.170988D+02	0.120967D-01	0.133064D+00
12	0.187402D+02	0.186044D+02	0.113233D-01	0.135880D+00
13	0.202433D+02	0.201050D+02	0.106356D-01	0.138263D+00
14	0.217417D+02	0.216014D+02	0.100217D-01	0.140303D+00
15	0.232361D+02	0.230941D+02	0.947111D-02	0.142067D+00
16	0.247270D+02	0.245834D+02	0.897530D-02	0.143605D+00
17	0.262148D+02	0.260698D+02	0.852691D-02	0.144958D+00
18	0.276999D+02	0.275537D+02	0.811982D-02	0.146157D+00
19	0.291825D+02	0.290353D+02	0.774882D-02	0.147228D+00
20	0.306631D+02	0.305149D+02	0.740951D-02	0.148190D+00
21	0.321417D+02	0.319927D+02	0.709812D-02	0.149060D+00
22	0.336187D+02	0.334689D+02	0.681143D-02	0.149852D+00
23	0.350941D+02	0.349436D+02	0.654669D-02	0.150574D+00
24	0.365682D+02	0.364170D+02	0.630152D-02	0.151236D+00
25	0.380410D+02	0.378891D+02	0.607384D-02	0.151846D+00
26	0.395125D+02	0.393601D+02	0.586188D-02	0.152409D+00
27	0.409830D+02	0.408301D+02	0.566408D-02	0.152930D+00
28	0.424525D+02	0.422991D+02	0.547907D-02	0.153414D+00
29	0.439210D+02	0.437672D+02	0.530567D-02	0.153865D+00
30	0.453887D+02	0.452344D+02	0.514282D-02	0.154285D+00
31	0.468554D+02	0.467008D+02	0.498960D-02	0.154678D+00
32	0.483214D+02	0.481664D+02	0.484517D-02	0.155045D+00
33	0.497867D+02	0.496313D+02	0.470881D-02	0.155391D+00
34	0.512513D+02	0.510955D+02	0.457987D-02	0.155716D+00
35	0.527152D+02	0.525591D+02	0.445776D-02	0.156022D+00
36	0.541785D+02	0.540221D+02	0.434195D-02	0.156310D+00
37	0.556412D+02	0.554846D+02	0.423198D-02	0.156583D+00
38	0.571033D+02	0.569465D+02	0.412741D-02	0.156842D+00
39	0.585650D+02	0.584079D+02	0.402786D-02	0.157087D+00
40	0.600261D+02	0.598688D+02	0.393298D-02	0.157319D+00
41	0.614868D+02	0.613293D+02	0.384245D-02	0.157541D+00
42	0.629471D+02	0.627893D+02	0.375598D-02	0.157751D+00
43	0.644069D+02	0.642489D+02	0.367330D-02	0.157952D+00
44	0.658663D+02	0.657082D+02	0.359418D-02	0.158144D+00
45	0.673254D+02	0.671671D+02	0.351838D-02	0.158327D+00
46	0.687841D+02	0.686256D+02	0.344570D-02	0.158502D+00
47	0.702425D+02	0.700838D+02	0.337596D-02	0.158670D+00
48	0.717005D+02	0.715417D+02	0.330898D-02	0.158831D+00
49	0.731582D+02	0.729992D+02	0.324460D-02	0.158986D+00
50	0.746156D+02	0.744565D+02	0.318268D-02	0.159134D+00
51	0.760728D+02	0.759135D+02	0.312306D-02	0.159276D+00
52	0.775296D+02	0.773702D+02	0.306564D-02	0.159413D+00
53	0.789862D+02	0.788257D+02	0.301029D-02	0.159545D+00
54	0.804425D+02	0.802829D+02	0.295689D-02	0.159672D+00
55	0.818986D+02	0.817388D+02	0.290536D-02	0.159795D+00
56	0.833545D+02	0.831946D+02	0.285558D-02	0.159913D+00
57	0.848101D+02	0.846500D+02	0.280748D-02	0.160027D+00
58	0.862654D+02	0.861053D+02	0.276098D-02	0.160137D+00
59	0.877206D+02	0.875604D+02	0.271598D-02	0.160243D+00
60	0.891756D+02	0.890152D+02	0.267243D-02	0.160346D+00
61	0.906303D+02	0.904699D+02	0.263024D-02	0.160445D+00
62	0.920848D+02	0.919243D+02	0.258937D-02	0.160541D+00
63	0.935392D+02	0.933786D+02	0.254974D-02	0.160634D+00
64	0.949934D+02	0.948326D+02	0.251131D-02	0.160724D+00
65	0.964473D+02	0.962865D+02	0.247402D-02	0.160811D+00
66	0.979011D+02	0.977403D+02	0.243782D-02	0.160896D+00
67	0.993548D+02	0.991938D+02	0.240266D-02	0.160978D+00
68	0.100808D+03	0.100647D+03	0.236849D-02	0.161057D+00
69	0.102262D+03	0.102100D+03	0.233529D-02	0.161135D+00
70	0.103715D+03	0.103553D+03	0.230300D-02	0.161210D+00

MAXNO = 5 MAX = .182834D - 01

MINO = 2 MIN = .111022D - 15

Table 2

defined in section 2 of this paper.

We have:

$$E\left\{ {}^{N}\Big/_{X_d \geq 1} \right\} = E\left\{ {}^{X_d}\Big/_{X_d \geq 1} \right\} = E\left\{ \frac{\lambda Y_a}{1 - e^{-\lambda Y_a}} \right\} \tag{47}$$

; where in (47) expression (17) has been used, and as in section 4 the conditioning $X_d \geq 1$ is deleted from expectations involving Y_a. Y_a is defined in section 2.

A simple transformation on the expectation $E\left\{\frac{\lambda Y_a}{1 - e^{-\lambda Y_a}}\right\}$ leads to the following equation:

$$E\left\{\frac{\lambda Y_a}{1 - e^{-\lambda Y_a}}\right\} = \lambda\ E\{Y_a\} + E\left\{\frac{\lambda Y_a}{e^{\lambda Y_a} - 1}\right\} \tag{48}$$

Using, as in section 4, the convexity of the function $\frac{\lambda x}{e^{\lambda x} - 1}$ for $x \geq 1$, and recalling that $Y_a \geq 1$, we obtain the following inequality from (48):

$$E\left\{\frac{\lambda Y_a}{1 - e^{-\lambda Y_a}}\right\} \leq \lambda\ E\{Y_a\} + \frac{\lambda}{e^{\lambda} - 1} \tag{49}$$

Substitution of (49) in (47) gives:

$$E\left\{ {}^{N}\Big/_{X_d \geq 1} \right\} \leq \lambda\ E\{Y_a\} + \frac{\lambda}{e^{\lambda} - 1} \tag{50}$$

Similarly, using the concavity of the logarithmic function, and the inequality (50), we obtain:

$$E\left\{ {}^{\log_2 N}\Big/_{X_d \geq 1} \right\} \leq \log_2 E\left\{ {}^{N}\Big/_{X_d \geq 1} \right\} \leq \log_2\left(\lambda\ E\{Y_a\} + \frac{\lambda}{e^{\lambda} - 1}\right) \tag{51}$$

Now recalling that L_N is the same with Y_d in section 2 of this paper, and using the bound (23) in section 4, we obtain:

$$E\left\{{}^{L_N}\Big/_{X_d \geq 1}\right\} = E\left\{{}^{Y_d}\Big/_{X_d \geq 1}\right\} \leq 2.8867\ \lambda\ E\{Y_a\} + \frac{\lambda}{e^{\lambda}-1} \tag{52}$$

Finally, using expressions (16) and (27) in section 4, we compute:

$$E\left\{{}^{\delta_{1N}}\Big/_{X_d \geq 1}\right\} = P\left({}^{X_d = 1}\Big/_{X_d \geq 1}\right) = E\left\{\frac{\lambda Y_a}{e^{\lambda Y_a}-1}\right\} \geq \lambda\ \frac{E\{Y_a\}}{e^{\lambda E\{Y_a\}}-1} \tag{53}$$

Through expressions (46), (50), (51), (52), and (53), we obtain the following upper bound on the expectation $E\{m_N\}$, where the conditioning $X_d \geq 1$ is now deleted as implicitly necessary:

$$\boxed{\begin{aligned} E\{m_N\} \leq\ & .5\ \ell og_2\left(\lambda\ E\{Y_a\} + \frac{\lambda}{e^{\lambda}-1}\right) + .5\ (2.8867\ \lambda\ E\{Y_a\} + \frac{\lambda}{e^{\lambda}-1}) \\ & + .019\ (\lambda\ E\{Y_a\} + \frac{\lambda}{e^{\lambda}-1}) - .519\ \lambda\ \frac{E\{Y_a\}}{e^{\lambda\ E\{Y_a\}}-1} \end{aligned}} \tag{54}$$

Due to difficulties in obtaining lower bounds for the expected value of the lower bound in (46), we will not search for a lower bound on the expectation $E\{m_N\}$. Instead, we will use the lower bound in (29) for the expected value $E\{D\}$. As we will see, our numerical results justify this approach.

The $E\{D\}$ and $E\{m_N\}$ expected values, are clearly related through the following equation:

$$E\{D\} = .5\ E\{Y_a\} + E\{m_N\} \tag{55}$$

Substitution of the bound in (54), in equation (55), provides the following upper bound on the expectation $E\{D\}$:

$$E\{D\} \leq .5\, \ell og_2 \left(\lambda\, E\{Y_a\} + \frac{\lambda}{e^{\lambda}-1}\right) + (.5 + 1.4434\, \lambda)\, E\{Y_a\} + \\ + \frac{\lambda}{2(e^{\lambda}-1)} + .019 \left(\lambda\, E\{Y_a\} + \frac{\lambda}{e^{\lambda}-1}\right) - .519\, \lambda\, \frac{E\{Y_a\}}{e^{\lambda E\{Y_a\}}-1} \tag{56}$$

Finally, using the bounds on $E\{Y_a\}$, as given by expressions (11) and (12) in section 3 of this paper, we obtain from (56) an upper bound on the expectation $E\{D\}$, which is a function of the rate λ only.

In table 3, we exhibit our computational results, where the upper bound is computed through expression (56), while the lower bound is computed through expression (29) and is the same as in table 1, section 4. Comparing the upper bounds in tables 1 and 3, we see that the bounds in table 3 are much tighter. In addition, we observe that for high λ values, close to the limit .3465 for stability of the CCRA, the lower and upper bounds in table 3 are very close to each other. This result justifies the fact that we maintained the same lower bound, as in section 4.

λ	Lower Bound	Upper Bound	Upper Bound From Table 1
0.0500	0.569	0.615	0.645
0.100	0.701	0.814	0.881
0.1500	0.961	1.193	1.319
0.1696	1.125	1.431	1.595
0.2000	1.505	1.979	2.236
0.2500	3.958	5.266	6.231
0.3000	9.521	11.889	15.096
0.3333	38.508	42.157	58.781
0.3400	80.643	85.119	122.043
0.3450	372.351	380.126	560.032

Table 3

6. References

[1] J. I. Capetanakis, "The Multiple Access Broadcast Channel: Protocol and Capacity Considerations," Ph.D. Thesis, Course VI, MIT, Cambridge, MA., August 1977.

[2] J. I. Capetanakis, "Tree Algorithms for Packet Broadcast Channels," IEEE Trans. Inf. Th. Vol. IT-25, pp. 505-515, September 1979.

[3] J. I. Capetanakis, "Generalized TDMA: The Multi-Accessing Tree Protocol," IEEE Trans. Comm. vol. COM-27, pp. 1476-1484, October 1979.

[4] J. L. Massey, "Collision-Resolution Algorithms and Random-Access Communications," UCLA, School of Engineering and Applied Science, Technical Report UCLA-ENG-8016, April 1980.

7. Appendix

Alternative approach to the computation of m_N

Define:

a_{jN} : The expected delay in the transmission of the jth successfully transmitted packet within some CRI, where N packets are involved in the collision within the first slot of the CRI.

Clearly, $a_{11} = 0$.

Now, let $N \geq 2$, and let the first trial event be (i, N-i). Then,

$$a_{jN} = \begin{cases} 1 + a_{ji} & ; \text{ for } j \leq i \leq N \\ 1 + L_i + a_{j-i,N-i} & ; \ 0 \leq i < j \leq N \end{cases} \tag{A.1}$$

Since the probability with which the event (i, N-i) occurs is $\binom{N}{i}\frac{1}{2^N}$, we obtain from (A.1):

$$2^N a_{jN} = \sum_{i=0}^{j-1}\binom{N}{i}(1 + L_i + a_{j-i,N-i}) + \sum_{i=j}^{N}\binom{N}{i}(1 + a_{ji}) =$$

$$= 2^N + \sum_{i=0}^{j-1}\binom{N}{i}L_i + \sum_{i=0}^{j-1}\binom{N}{i}a_{j-i,N-i} + \sum_{i=j}^{N}\binom{N}{i}a_{ji} \tag{A.2}$$

Applying some transformations on (A.2), we finally obtain the following recursive expressions for the a_{jk} parameters:

$$a_{11} = 0,\ a_{12} = 2.5,\ a_{22} = 3.5$$

$$(2^N-2)a_{1N} = 2^N + 1 + \sum_{i=1}^{N} \binom{N}{i} a_{1i} \quad ; \quad N \geq 2$$

$$(2^N-2)a_{jN} = 2^N + \sum_{i=0}^{j-1} \binom{N}{i} L_i + \sum_{i=1}^{j-1} \binom{N}{i} a_{j-i,N-i} + \sum_{i=j}^{N-1} \binom{N}{i} a_{ji} \quad ; \quad 2 \leq j \leq N-1, \quad N \geq 2 \tag{A.3}$$

$$(2^N-2)a_{NN} = 2^N + \sum_{i=0}^{N-1} \binom{N}{i} L_i + \sum_{i=1}^{N-1} \binom{N}{i} a_{ii} \quad ; \quad N \geq 2$$

Each of the N packets is the ith successfully transmitted packet within the CRI, with equal probabilities for all $i : 1 \leq i \leq N$. Therefore,

$$m_N = N^{-1} \sum_{i=1}^{N} a_{iN} \tag{A.4}$$

By substituting expressions (A.3) in (A.4) and applying some transformations, we obtain the expressions in (37).

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